Proof by Induction

Questions

Q1.

(i) A sequence of numbers is defined by

$$u_1 = 6, \qquad u_2 = 27$$

$$u_{n+2} = 6u_{n+1} - 9u_n \qquad n \geq 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 3^n(n + 1)$$

(6)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 3^{3n-2} + 2^{3n+1} \text{ is divisible by } 19$$

(6)

(Total for question = 12 marks)

Q2.

Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

(Total for question = 6 marks)

Q3.

Prove by induction that for all positive integers *n*

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

(Total for question = 6 marks)

Q4.

Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 2^{n+2} + 3^{2n+1}$$

is divisible by 7

(Total for question = 6 marks)

Q5.

(a) Prove by induction that, for all positive integers *n*,

$$\sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$$

(6)

(b) Hence, show that, for all positive integers *n*,

$$\sum_{r=n}^{2n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)(an+b)(cn+d)$$

where *a*, *b*, *c* and *d* are integers to be determined.

(3)

(Total for question = 9 marks)

Q6.

(i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix}$$

(6)

(ii) Prove by induction that for $n \in \mathbb{Z}^+$

$$f(n) = 4^{n+1} + 5^{2n-1}$$

is divisible by 21

(6)

(Total for question = 12 marks)

Q7.

Prove by induction that for all positive integers *n*,

$$f(n) = 2^{3n+1} + 3(5^{2n+1})$$

is divisible by 17

(6)

(Total for question = 6 marks)

Q8.

(a) Prove by induction that for all positive integers n,

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

(6)

(b) Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to show that for all positive integers n,

$$\sum_{r=1}^{n} r(r+6)(r-6) = \frac{1}{4}n(n+1)(n-8)(n+9)$$

(4)

(c) Hence find the value of n that satisfies

$$\sum_{r=1}^{n} r(r+6)(r-6) = 17\sum_{r=1}^{n} r^2$$

(5)

(Total for question = 15 marks)

Q9.

(i)

$$\mathbf{M} = \begin{pmatrix} 2 & a & 4 \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

where a is a constant.

(a) For which values of a does the matrix **M** have an inverse?

(2)

Given that **M** is non-singular,

(b) find $\mathbf{M}^{-1}$ in terms of a

(4)

(ii) Prove by induction that for all positive integers n,

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$$

(6)

(Total for question = 12 marks)

Mark Scheme – Proof by Induction

Q1.

Question Number	Scheme		Marks
	$u_{n+2} = 6u_{n+1} - 9u_n,\ n \geqslant 1,\ u_1 = 6,\ u_2 = 27;\quad u_n = 3^n(n+1)$		
(i)	$n=1;\quad u_1 = 3(2) = 6$	Check that $u_1 = 6$ and $u_2 = 27$	B1
	$n=2;\quad u_2 = 3^2(2+1) = 27$		
	So u_n is true when $n=1$ and $n=2$.		
	Assume that $u_k = 3^k(k+1)$ and $u_{k+1} = 3^{k+1}(k+2)$ are true.	Could assume for $n=k, n=k-1$ and show for $n=k+1$	
	Then $u_{k+2} = 6u_{k+1} - 9u_k$		
	$= 6(3^{k+1})(k+2) - 9(3^k)(k+1)$	Substituting u_k and u_{k+1} into $u_{k+2} = 6u_{k+1} - 9u_k$	M1
		Correct expression	A1
	$= 2(3^{k+2})(k+2) - (3^{k+2})(k+1)$	Achieves an expression in 3^{k+2}	M1
	$= (3^{k+2})(2k+4-k-1)$		
	$= (3^{k+2})(k+3)$		
	$= (3^{k+2})(k+2+1)$	$(3^{k+2})(k+2+1)$ or $(3^{k+2})(k+3)$	A1
	If the result is **true for $n = k$ and $n = k+1$** then it is now **true for $n = k+2$**. As it is **true for $n = 1$ and $n = 2$** then it is **true for all n** $(\in \mathbb{Z}^+)$.	Correct conclusion seen at the end. Condone true for $n = 1$ and $n = 2$ seen anywhere. This should be compatible with assumptions.	A1 cso
			[6]
(ii)	$f(n) = 3^{3n-2} + 2^{3n+1}$ is divisible by 19		
	In all ways, first M is for applying $f(k+1)$ with at least 1 power correct. The second M is dependent on at least one accuracy being awarded and making $f(k+1)$ the subject and the final A is correct solution only.		
(ii) **Way 1**	$f(1) = 3^1 + 2^4 = 19$ {which is divisible by 19}.	Shows $f(1) = 19$	B1
	$\{\therefore f(n)$ is divisible by 19 when $n = 1\}$		
	{Assume that for $n = k$,		
	$f(k) = 3^{3k-2} + 2^{3k+1}$ is divisible by 19 for $k \in \mathbb{Z}^+$.}		
	$f(k+1) - f(k) = 3^{3(k+1)-2} + 2^{3(k+1)+1} - (3^{3k-2} + 2^{3k+1})$	Applies $f(k+1)$ with at least 1 power correct	M1
	$f(k+1) - f(k) = 27(3^{3k-2}) + 8(2^{3k+1}) - (3^{3k-2} + 2^{3k+1})$		
	$f(k+1) - f(k) = 26(3^{3k-2}) + 7(2^{3k+1})$		
	$= 7(3^{3k-2} + 2^{3k+1}) + 19(3^{3k-2})$ Either	$7(3^{3k-2} + 2^{3k+1})$ or $7f(k)$; $19(3^{3k-2})$	A1;
	or $= 26(3^{3k-2} + 2^{3k+1}) - 19(2^{3k+1})$ or	$26(3^{3k-2} + 2^{3k+1})$ or $26f(k)$; $-19(2^{3k+1})$	A1
	$= 7f(k) + 19(3^{3k-2})$		
	or $= 26f(k) - 19(2^{3k+1})$		
	$\therefore f(k+1) = 8f(k) + 19(3^{3k-2})$	Dependent on at least one of the previous accuracy marks being awarded.	dM1
	or $f(k+1) = 27f(k) - 19(2^{3k+1})$	Makes Applies $f(k+1)$ with at least 1 power correct the subject	
	$\{\therefore f(k+1) = 8f(k) + 19(3^{3k-2})$ is divisible by 19 as both		
	$8f(k)$ and $19(3^{3k-2})$ are both divisible by 19}		

	If the result is **true for $n=k$**, then it is now **true for $n=k+1$**. As the result has shown to be **true for $n=1$**, then the result is **true for all n** $(\in\mathbb{Z}^+)$.	Correct conclusion seen at the end. Condone true for n = 1 stated earlier.	A1 cso **[6]**
(ii) **Way 2**	$\mathrm{f}(1)=3^1+2^4=19$ {which is divisible by 19}. $\{\therefore \mathrm{f}(n)$ is divisible by 19 when $n=1\}$	Shows $\mathrm{f}(1)=19$	B1
	Assume that for $n=k$, $\mathrm{f}(k)=3^{3k-2}+2^{3k+1}$ is divisible by 19 for $k\in\mathbb{Z}^+$.		
	$\mathrm{f}(k+1)=3^{3(k+1)-2}+2^{3(k+1)+1}$	Applies $\mathrm{f}(k+1)$ with at least 1 power correct	M1
	$\mathrm{f}(k+1)=27(3^{3k-2})+8(2^{3k+1})$ $=8(3^{3k-2}+2^{3k+1})+19(3^{3k-2})$ Either **or** $=27(3^{3k-2}+2^{3k+1})-19(2^{3k+1})$ or	$8(3^{3k-2}+2^{3k+1})$ or $8\mathrm{f}(k)$; $19(3^{3k-2})$ $27(3^{3k-2}+2^{3k+1})$ or $27\mathrm{f}(k)$; $-19(2^{3k+1})$	A1; A1
	$\therefore \mathrm{f}(k+1)=8\mathrm{f}(k)+19(3^{3k-2})$ **or** $\mathrm{f}(k+1)=27\mathrm{f}(k)-19(2^{3k+1})$	**Dependent on at least one of the previous accuracy marks being awarded.**	dM1
	$\{\therefore \mathrm{f}(k+1)=8\mathrm{f}(k)+19(3^{3k-2})$ is divisible by 19 as both $8\mathrm{f}(k)$ and $19(3^{3k-2})$ are both divisible by 19}		
	If the result is **true for $n=k$**, then it is now **true for $n=k+1$**. As the result has shown to be **true for $n=1$**, then the result is **true for all n** $(\in\mathbb{Z}^+)$.	Correct conclusion seen at the end. Condone true for n = 1 stated earlier.	A1 cso **[6]**
(ii) **Way 3**	$\mathrm{f}(n)=3^{3n-2}+2^{3n+1}$ is divisible by 19 $\mathrm{f}(1)=3^1+2^4=19$ {which is divisible by 19}. $\{\therefore \mathrm{f}(n)$ is divisible by 19 when $n=1\}$	Shows $\mathrm{f}(1)=19$	B1
	Assume that for $n=k$, $\mathrm{f}(k)=3^{3k-2}+2^{3k+1}$ is divisible by 19 for $k\in\mathbb{Z}^+$.		
	$\mathrm{f}(k+1)-\alpha\mathrm{f}(k)=3^{3(k+1)-2}+2^{3(k+1)+1}-\alpha(3^{3k-2}+2^{3k+1})$	Applies $\mathrm{f}(k+1)$ with at least 1 power correct	M1
	$\mathrm{f}(k+1)-\alpha\mathrm{f}(k)=(27-\alpha)(3^{3k-2})+(8-\alpha)2^{3k+1}$ $=(8-\alpha)(3^{3k-2}+2^{3k+1})+19(3^{3k-2})$ **or** $=(27-\alpha)(3^{3k-2}+2^{3k+1})-19(2^{3k+1})$	$(8-\alpha)(3^{3k-2}+2^{3k+1})$ or $(8-\alpha)\mathrm{f}(k)$; $19(3^{3k-2})$ NB choosing $\alpha=8$ makes first term disappear. $(27-\alpha)(3^{3k-2}+2^{3k+1})$ or $(27-\alpha)\mathrm{f}(k)$; $-19(2^{3k+1})$ NB choosing $\alpha=27$ makes first term disappear.	A1; A1
	$\therefore \mathrm{f}(k+1)=8\mathrm{f}(k)+19(3^{3k-2})$ **or** $\mathrm{f}(k+1)=27\mathrm{f}(k)-19(2^{3k+1})$	**Dependent on at least one of the previous accuracy marks being awarded.** Makes $\mathrm{f}(k+1)$ the subject.	dM1
	$\{\therefore \mathrm{f}(k+1)=27\mathrm{f}(k)-19(2^{3k+1})$ is divisible by 19 as both $27\mathrm{f}(k)$ and $19(2^{3k+1})$ are both divisible by 19}		
	If the result is **true for $n=k$**, then it is now **true for $n=k+1$**. As the result has shown to be **true for $n=1$**, then the result is true **for all n** $(\in\mathbb{Z}^+)$.	Correct conclusion seen at the end. Condone true for n = 1 stated earlier.	A1 cso **[6]** **12**
	Question Notes		
(ii)	Accept use of $\mathrm{f}(k)=3^{3k-2}+2^{3k+1}=19m$ o.e. and award method and accuracy as above.		

Q2.

Question	Scheme	Marks	AOs
	$n=1$, $\sum_{r=1}^{1}\frac{1}{(2r-1)(2r+1)}=\frac{1}{1\times 3}=\frac{1}{3}$ and $\frac{n}{2n+1}=\frac{1}{2\times 1+1}=\frac{1}{3}$ (true for n=1)	B1	2.2a
	Assume general statement is true for $n=k$. So assume $\sum_{r=1}^{k}\frac{1}{(2r-1)(2r+1)}=\frac{k}{2k+1}$ is true.	M1	2.4
	$\left(\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\right)$ "$\frac{k}{2k+1}$" $+\frac{1}{(2k+1)(2k+3)}$	M1	2.1
	$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$	dM1	1.1b
	$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}=\frac{(k+1)}{2(k+1)+1}$ or $\frac{k+1}{2k+3}$	A1	1.1b
	As $\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\frac{(k+1)}{2(k+1)+1}$ then the general result is true for $n=k+1$ As the general result has been shown to be <u>true for $n=1$</u>, and <u>true for $n=k$ implies true for $n=k+1$</u>, so the result <u>is true for all n</u> $\in\mathbb{N}$	A1cso	2.4
		(6)	
			(6 marks)

Notes		
	B1	Substitutes $n=1$ into both sides of the statement to show they are equal. As a minimum expect to see $\frac{1}{1\times 3}$ and $\frac{1}{2+1}$ for the substitutions. (No need to state true for $n=1$ for this mark.)
	M1	Assumes (general result) true for $n=k$. (Assume (true for) $n=k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n=k$ then ... etc.)
	M1	Attempts to add $(k+1)$th term to their sum of k terms. Must be adding the $(k+1)$th term but allow slips with the sum.
	dM1	Depends on previous M. Combines their two fractions over a correct common denominator for their fractions, which may be $(2k+1)^2(2k+3)$ (allow a slip in the numerator).
	A1	Correct algebraic work leading to $\frac{(k+1)}{2(k+1)+1}$ or $\frac{k+1}{2k+3}$
	A1	cso Depends on all except the B mark being scored (but must have an attempt to show the $n=1$ case). Demonstrates the expression is the correct for $n=k+1$ (both sides must have been seen somewhere) and gives a correct induction statement with **all** three underlined statements (or equivalents) seen at some stage during their solution (so true for $n=1$ may be seen at the start). For demonstrating the correct expression, accept giving in the form $\frac{(k+1)}{2(k+1)+1}$, or reaching $\frac{k+1}{2k+3}$ and stating "which is the correct form with $n=k+1$" or similar – but some indication is needed. Note: if mixed variables are used in working (r's and k's mixed up) then withhold the final A. Note: If n is used throughout instead of k allow all marks if earned.

Q3.

Question	Scheme	Marks	AOs
	Way 1 f(k+1) – f(k)		
	When $n = 1$, $3^{2n+4} - 2^{2n} = 729 - 4 = 725$ ($725 = 145 \times 5$) so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $3^{2k+4} - 2^{2k}$ is divisible by 5	M1	2.4
	$\mathrm{f}(k+1) - \mathrm{f}(k) = 3^{2k+6} - 2^{2k+2} - 3^{2k+4} + 2^{2k}$	M1	2.1
	either $8\mathrm{f}(k) + 5 \times 2^{2k}$ or $3\mathrm{f}(k) + 5 \times 3^{2k+4}$	A1	1.1b
	$\mathrm{f}(k+1) = 9\mathrm{f}(k) + 5 \times 2^{2k}$ or $\mathrm{f}(k+1) = 4\mathrm{f}(k) + 5 \times 3^{2k+4}$ o.e.	A1	1.1b
	If true for $n = k$ then it is true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all (positive integers) n. (Allow 'for all values')	A1	2.4
		(6)	

Way 2 $f(k+1)$		
When $n=1$, $3^{2n+4}-2^{2n}=729-4=725$ ($725=145\times5$) so the statement is true for $n=1$	B1	2.2a
Assume true for $n=k$ so $3^{2k+4}-2^{2k}$ is divisible by 5	M1	2.4
$f(k+1)=3^{2(k+1)+4}-2^{2(k+1)}\left(=3^{2k+6}-2^{2k+2}\right)$	M1	2.1
$f(k+1)=9f(k)+5\times2^{2k}$ or $f(k+1)=4f(k)+5\times3^{2k+4}$ o.e.	A1 A1	1.1b 1.1b
If true for $n=k$ then it is true for $n=k+1$ and as it is true for $n=1$, the statement is true for all (positive integers) n. (Allow 'for all values')	A1	2.4
	(6)	
Way 3 $f(k)=5M$		
When $n=1$, $3^{2n+4}-2^{2n}=729-4=725$ ($725=145\times5$) so the statement is true for $n=1$	B1	2.2a
Assume true for $n=k$ so $3^{2k+4}-2^{2k}=5M$	M1	2.4
$f(k+1)=3^{2(k+1)+4}-2^{2(k+1)}\left(=3^{2k+6}-2^{2k+2}\right)$	M1	2.1
$\left(f(k+1)=3^2\times3^{2k+4}-2^2\times2^{2k}=3^2\times\left(5M+2^{2k+2}\right)-2^2\times2^{2k}\right)$ $f(k+1)=45M+5\times2^{2k}$ o.e. OR $\left(f(k+1)=3^2\times3^{2k+4}-2^2\times2^{2k}=3^2\times3^{2k+4}-2^2\times\left(3^{2k+4}-5M\right)\right)$ $f(k+1)=5\times3^{2k+4}+20M$ o.e.	A1 A1	1.1b 1.1b
If true for $n=k$ then it is true for $n=k+1$ and as it is true for $n=1$, the statement is true for all (positive integers) n. (Allow 'for all values')	A1	2.4
	(6)	

	Way 4 $f(k+1)+f(k)$		
	When $n=1$, $3^{2n+4}-2^{2n}=729-4=725$ ($725=145\times5$) so the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$ so $3^{2k+4}-2^{2k}$ is divisible by 5	M1	2.4
	$f(k+1)+f(k)=3^{2k+6}-2^{2k+2}+3^{2k+4}-2^{2k}$	M1	2.1
	$f(k+1)+f(k)=3^2\times3^{2k+4}-2^2\times2^{2k}+3^{2k+4}-2^{2k}$	A1	1.1b
	leading to $10\times3^{2k+4}-5\times2^{2k}$		
	$f(k+1)=5\left[2\times3^{2k+4}-2^{2k}\right]-f(k)$ o.e.	A1	1.1b
	If true for $n=k$ then it is true for $n=k+1$ and as it is true for $n=1$, the statement is true for all (positive integers) n. (Allow 'for all values')	A1	2.4
		(6)	

	Way 5 $f(k+1)-$'M'$f(k)$ (Selecting a value of M that will lead to multiples of 5)		
	When $n=1$, $3^{2n+4}-2^{2n}=729-4=725$ ($725=145\times5$) so the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$ so $3^{2k+4}-2^{2k}$ is divisible by 5	M1	2.4
	$f(k+1)-\text{'M'}f(k)=3^{2k+6}-2^{2k+2}-\text{'M'}\times3^{2k+4}+\text{'M'}\times2^{2k}$	M1	2.1
	$f(k+1)-\text{'M'}f(k)=(9-\text{'M'})\times3^{2k+4}-(4-\text{'M'})\times2^{2k}$	A1	1.1b
	$f(k+1)=(9-\text{'M'})\times3^{2k+4}-(4-\text{'M'})\times2^{2k}+\text{'M'}f(k)$ o.e.	A1	1.1b
	If true for $n=k$ then it is true for $n=k+1$ and as it is true for $n=1$, the statement is true for all (positive integers) n. (Allow 'for all values')	A1	2.4
		(6)	
			(6 marks)

Notes

<u>Way 1</u> f(k +1) – f(k)

B1: Shows the statement is true for $n = 1$. Needs to show $f(1) = 725$ and conclusion true for $n = 1$, this statement can be recovered in their conclusion if says e.g. true for $n = 1$

M1: Makes an assumption statement that assumes the result is true for $n = k$. Assume (true for) $n = k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n = k$ then …etc

M1: Attempts f(k +1) – f(k) or equivalent work

A1: Achieves a correct simplified expression for f(k +1) – f(k)

A1: Achieves a correct expression for f(k +1) in terms of f(k)

A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

<u>Way 2</u> f(k +1)

B1: Shows the statement is true for $n = 1$. Needs to show $f(1) = 725$ and conclusion true for $n = 1$, this statement can be recovered in their conclusion if says e.g. true for $n = 1$.

M1: Makes an assumption statement that assumes the result is true for $n = k$. Assume (true for) $n = k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n = k$ then …etc

M1: Attempts f(k +1)

A1: Correctly achieves either $9\text{f}\ k$ **or** 5×2^{2k} or either $4\text{f}\ k$ **or** $5\times 3^{2k+4}$

A1: Achieves a correct expression for f(k +1) in terms of f(k)

A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

<u>Way 3</u> f(k) = 5M

B1: Shows the statement is true for $n = 1$. Needs to show $f(1) = 725$ and conclusion true for $n = 1$, this statement can be recovered in their conclusion if says e.g. true for $n = 1$.

M1: Makes an assumption statement that assumes the result is true for $n = k$. Assume (true for) $n = k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n = k$ then …etc

M1: Attempts f(k +1)

A1: Correctly achieves either $45M$ **or** 5×2^{2k} or either $20M$ **or** $5\times 3^{2k+4}$

A1: Achieves a correct expression for f(k +1) in terms of M and 2^{2k} or M and 3^{2k+4}

A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

<u>Way 4</u> f(k +1) + f(k)

B1: Shows the statement is true for $n = 1$. Needs to show $f(1) = 725$ and conclusion true for $n = 1$, this statement can be recovered in their conclusion if says e.g. true for $n = 1$

M1: Makes an assumption statement that assumes the result is true for $n = k$. Assume (true for) $n = k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n = k$ then …etc

M1: Attempts f(k +1) + f(k) or equivalent work

A1: Achieves a correct simplified expression for f(k +1) + f(k)

A1: Achieves a correct expression for $\text{f}\ k+1\ = 5\left[2\times 3^{2k+4} - 2^{2k}\right] - \text{f}(k)$

A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

Way 5 f(k +1) – Mf(k) (Selects a suitable value for M which leads to divisibility of 5)
B1: Shows the statement is true for $n = 1$. Needs to show $f(1) = 725$ and conclusion true for $n = 1$, this statement can be recovered in their conclusion if says e.g. true for $n = 1$
M1: Makes an assumption statement that assumes the result is true for $n = k$. Assume (true for) $n = k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n = k$ then …etc
M1: Attempts f(k +1) – Mf(k) or equivalent work
A1: Achieves a correct simplified expression, $f(k+1) - 'M'f(k)$ which is divisible by 5

$f(k+1) - 'M'f(k) = (9 - 'M') \times 3^{2k+4} - (4 - 'M') \times 2^{2k}$

A1: Achieves a correct expression for $f(k+1) = (9 - 'M') \times 3^{2k+4} - (4 - 'M') \times 2^{2k} + 'M'f(k)$ which is divisible by 5
A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

Q4.

Question	Scheme	Marks	AOs
	Way 1: $\mathbf{f}(k+1)-\mathbf{f}(k)$ When $n=1$, $2^{n+2}+3^{2n+1}=2^3+3^3=35$ Shows the statement is true for $n=1$, allow 5(7)	B1	2.2a
	Assume true for $n=k$, so $2^{k+2}+3^{2k+1}$ is divisible by 7	M1	2.4
	$f(k+1)-f(k)=2^{k+3}+3^{2k+3}-(2^{k+2}+3^{2k+1})$	M1	2.1
	$=2\times2^{k+2}+9\times3^{2k+1}-2^{k+2}-3^{2k+1}$ $=2^{k+2}+8\times3^{2k+1}$ $=f(k)+7\times3^{2k+1}$ or $8f(k)-7\times2^{k+2}$	A1	1.1b
	$f(k+1)=2f(k)+7\times3^{2k+1}$ or $9f(k)-7\times2^{k+2}$	A1	1.1b
	If true for $n=k$ then true for $n=k+1$ and as it is **true for $n=1$** the statement is **true for all** (positive integers) n	A1	2.4
		(6)	
	Way 2: $\mathbf{f}(k+1)$ When $n=1$, $2^{n+2}+3^{2n+1}=2^3+3^3=35$ So the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$, so $2^{k+2}+3^{2k+1}$ is divisible by 7	M1	2.4
	$f(k+1)=2^{(k+1)+2}+3^{2(k+1)+1}$	M1	2.1
	$f(k+1)=2^{k+3}+3^{2k+3}=2\times2^{k+2}+9\times3^{2k+1}$ $=2(2^{k+2}+3^{2k+1})+7\times3^{2k+1}$ $=2f(k)+7\times3^{2k+1}$ or $9f(k)-7\times2^{k+2}$	A1 A1	1.1b 1.1b
	If true for $n=k$ then true for $n=k+1$ and as it is **true for $n=1$** the statement is **true for all** (positive integers) n	A1	2.4
		(6)	

	Way 3: $\mathbf{f}(k+1)-m\,\mathbf{f}(k)$ When $n=1$, $2^{n+2}+3^{2n+1}=2^3+3^3=35$ So the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$, so $2^{k+2}+3^{2k+1}$ is divisible by 7	M1	2.4
	$f(k+1)-mf(k)=2^{k+3}+3^{2k+3}-m\left(2^{k+2}+3^{2k+1}\right)$	M1	2.1
	$=2\times2^{k+2}+9\times3^{2k+1}-m\times2^{k+2}-m\times3^{2k+1}$ $=(2-m)2^{k+2}+9\times3^{2k+1}-m\times3^{2k+1}$ $=(2-m)\left(2^{k+2}+3^{2k+1}\right)+7\times3^{2k+1}$	A1	1.1b
	$f(k+1)=(2-m)\left(2^{k+2}+3^{2k+1}\right)+7\times3^{2k+1}+mf(k)$	A1	1.1b
	If true for $n=k$ **then true for** $n=k+1$ and as it is **true for** $n=1$ the statement is **true for all** (positive integers) n	A1	2.4
		(6)	
			(6 marks)

Notes:

Way 1: $\mathbf{f}(k+1)-\mathbf{f}(k)$

B1: Shows that f(1) = 35 and concludes or shows divisible by 7. This may be seen in the final statement.

M1: Makes a statement that assumes the result is true for some value of n

M1: Attempts $f(k+1)-f(k)$

A1: Achieves a correct expression for $f(k+1)-f(k)$ in terms of $f(k)$

A1: Reaches a correct expression for $f(k+1)$ in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks, look out for scoring B1 in this statement. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution.

Way 2: $\mathbf{f}(k+1)$

B1: Shows that f(1) = 35 and concludes divisible by 7

M1: Makes a statement that assumes the result is true for some value of n

M1: Attempts $f(k+1)$

A1: Correctly obtains either $2f(k)$ or $7\times3^{2k+1}$ or either $9f(k)$ or $-7\times2^{k+2}$

A1: Reaches a correct expression for $f(k+1)$ in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution.

Way 3: $\mathbf{f}(k+1)-m\,\mathbf{f}(k)$

B1: Shows that f(1) = 35 and concludes divisible by 7

M1: Makes a statement that assumes the result is true for some value of n

M1: Attempts $f(k+1)-mf(k)$

A1: Achieves a correct expression for $f(k+1)-mf(k)$ in terms of $f(k)$

A1: Reaches a correct expression for $f(k+1)$ in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution.

Q5.

Question	Scheme	Marks	AOs
(a)	$n=1$, $\text{lhs}=1(2)(3)=6$, $\text{rhs}=\frac{1}{2}(1)(2)^2(3)=6$ (true for $n = 1$)	B1	2.2a
	Assume true for $n = k$ so $\sum_{r=1}^{k} r(r+1)(2r+1)=\frac{1}{2}k(k+1)^2(k+2)$	M1	2.4
	$\sum_{r=1}^{k+1} r(r+1)(2r+1)=\frac{1}{2}k(k+1)^2(k+2)+(k+1)(k+2)(2k+3)$	M1	2.1
	$=\frac{1}{2}(k+1)(k+2)\left[k(k+1)+2(2k+3)\right]$	dM1	1.1b
	$=\frac{1}{2}(k+1)(k+2)\left[k^2+5k+6\right]=\frac{1}{2}(k+1)(k+2)(k+2)(k+3)$ Shows that $=\frac{1}{2}(\underline{k+1})(\underline{k+1}+1)^2(\underline{k+1}+2)$ Alternatively shows that $\sum_{r=1}^{k+1} r(r+1)(2r+1)=\frac{1}{2}(k+1)(k+1+1)^2(k+1+2)$ $=\frac{1}{2}(k+1)(k+2)^2(k+3)$ Compares with their summation and concludes true for $n=k+1$, may be seen in the conclusion.	A1	1.1b
	If the statement is **true for** $n = k$ **then** it has been shown **true for** $n = k + 1$ and as it is **true for** $n = 1$, the statement is **true** for all **positive integers** n.	A1	2.4
		(6)	
(b)	$\sum_{r=n}^{2n} r(r+1)(2r+1)=\frac{1}{2}(2n)(2n+1)^2(2n+2)-\frac{1}{2}(n-1)n^2(n+1)$	M1	3.1a
	$=\frac{1}{2}n(n+1)\left[4(2n+1)^2-n(n-1)\right]$	M1	1.1b
	$=\frac{1}{2}n(n+1)(15n^2+17n+4)$ $=\frac{1}{2}n(n+1)(3n+1)(5n+4)$	A1	1.1b
		(3)	
			(9 marks)

Notes
(a) Note ePen B1 M1 M1 A1 A1 A1 B1: Substitutes $n = 1$ into both sides to show that they are both equal to 6. (There is no need to state true for $n = 1$ for this mark) M1: Makes a statement that assumes the result is true for some value of n, say k M1: Adds the $(k + 1)$th term to the assumed result dM1: Dependent on previous M, factorises out $\frac{1}{2}(k+1)(k+2)$ A1: Reaches a correct the required expression no errors and shows that this is the correct sum for $n = k+1$ A1: Depends on all except **B** mark being scored (must have been some attempt to show true for $n = 1$). Correct conclusion conveying all the points in bold. (b) M1: Realises that $\sum_{r=1}^{2n} r(r+1)(2r+1) - \sum_{r=1}^{n-1} r(r+1)(2r+1)$ is required and uses the result from part (a) to obtain the required sum in terms of n M1: Attempts to factorise by $\frac{1}{2}n(n+1)$ A1: Correct expression or correct values

Q6.

Question	Scheme	Marks	AOs
(i)	$n=1, \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^1 = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}, \quad \begin{pmatrix} 4\times1+1 & -8(1) \\ 2\times1 & 1-4(1) \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$ So the result is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k = \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}$	M1	2.4
	$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^{k+1} = \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$ or $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^{k+1} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}$	M1	1.1b
	$\begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 5(4k+1)-16k & -8(4k+1)+24k \\ 10k+2(1-4k) & -16k-3(1-4k) \end{pmatrix}$ or $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix} = \begin{pmatrix} 5(4k+1)-16k & -40k-8(1-4k) \\ 2(1+4k)-6k & -16k-3(1-4k) \end{pmatrix}$	A1	1.1b
	$= \begin{pmatrix} 4(k+1)+1 & -8(k+1) \\ 2(k+1) & 1-4(k+1) \end{pmatrix}$	A1	2.1
	<u>If true for $n = k$</u> then <u>true for $n = k + 1$</u>, <u>true for $n = 1$</u> so <u>true for all (positive integers) n</u> (Allow "for all values")	A1	2.4
		(6)	

(ii) **Way 1**	$f(k+1)-f(k)$		
	When $n=1$, $4^{n+1}+5^{2n-1}=16+5=21$ so the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$ so $4^{k+1}+5^{2k-1}$ is divisible by 21	M1	2.4
	$f(k+1)-f(k)=4^{k+2}+5^{2k+1}-4^{k+1}-5^{2k-1}$	M1	2.1
	$=4\times4^{k+1}+25\times5^{2k-1}-4^{k+1}-5^{2k-1}$		
	$=3f(k)+21\times5^{2k-1}$ or e.g. $=24f(k)-21\times4^{k+1}$	A1	1.1b
	$f(k+1)=4f(k)+21\times5^{2k-1}$ or e.g. $f(k+1)=25f(k)-21\times4^{k+1}$	A1	1.1b
	If true for $n=k$ then true for $n=k+1$, true for $n=1$ so true for all (positive integers) n (Allow "for all values")	A1	2.4
		(6)	

(ii) **Way 2**	$f(k+1)$		
	When $n=1$, $4^{n+1}+5^{2n-1}=16+5=21$ so the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$ so $4^{k+1}+5^{2k-1}$ is divisible by 21	M1	2.4
	$f(k+1)=4^{k+1+1}+5^{2(k+1)-1}$	M1	2.1
	$f(k+1)=4\times4^{k+1}+5^{2k+1}=4\times4^{k+1}+4\times5^{2k-1}+25\times5^{2k-1}-4\times5^{2k-1}$ $f(k+1)=4f(k)+21\times5^{2k-1}$	A1 A1	1.1b 1.1b
	If true for $n=k$ then true for $n=k+1$, true for $n=1$ so true for all (positive integers) n (Allow "for all values")	A1	2.4
		(6)	
(ii) **Way 3**	$f(k+1)-mf(k)$		
	When $n=1$, $4^{n+1}+5^{2n-1}=16+5=21$ so the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$ so $4^{k+1}+5^{2k-1}$ is divisible by 21	M1	2.4
	$f(k+1)-mf(k)=4^{k+2}+5^{2k+1}-m(4^{k+1}+5^{2k-1})$	M1	2.1
	$=(4-m)4^{k+1}+5^{2k+1}-m\times5^{2k-1}$ $=(4-m)(4^{k+1}+5^{2k-1})+21\times5^{2k-1}$	A1	1.1b
	$=(4-m)(4^{k+1}+5^{2k-1})+21\times5^{2k-1}+mf(k)$	A1	1.1b
	If true for $n=k$ then true for $n=k+1$, true for $n=1$ so true for all (positive integers) n (Allow "for all values")	A1	2.4
		(6)	
(ii) **Way 4**	$f(k)=21M$		
	When $n=1$, $4^{n+1}+5^{2n-1}=16+5=21$ so the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$ so $4^{k+1}+5^{2k-1}=21M$	M1	2.4
	$f(k+1)=4^{k+1+1}+5^{2(k+1)-1}$	M1	2.1
	$f(k+1)=4\times4^{k+1}+5^{2k+1}=4(21M-5^{2k-1})+5^{2k+1}$ $f(k+1)=84M+21\times5^{2k-1}$	A1 A1	1.1b 1.1b
	If true for $n=k$ then true for $n=k+1$, true for $n=1$ so true for all (positive integers) n (Allow "for all values")	A1	2.4
		(6)	
			(12 marks)

Notes
(i) B1: Shows that the result holds for $n = 1$. Must see **substitution** into the rhs. The minimum would be: $\begin{pmatrix} 4+1 & -8 \\ 2 & 1-4 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$. M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.) M1: Sets up a correct multiplication statement either way round A1: Achieves a correct un-simplified matrix A1: Reaches a correct simplified matrix with no errors **and the correct un-simplified matrix seen previously.** Note that the simplified result may be proved by equivalence. A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution. (ii) **Way 1** B1: Shows that f(1) = 21 M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.) M1: Attempts $f(k + 1) – f(k)$ or equivalent work A1: Achieves a correct expression for $f(k + 1) – f(k)$ in terms of $f(k)$ A1: Reaches a correct expression for $f(k + 1)$ in terms of $f(k)$ A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.

Way 2 B1: Shows that f(1) = 21 M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.) M1: Attempts $f(k + 1)$ A1: Correctly obtains $4f(k)$ **or** $21\times 5^{2k-1}$ A1: Reaches a correct expression for $f(k + 1)$ in terms of $f(k)$ A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution. **Way 3** B1: Shows that f(1) = 21 M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.) M1: Attempts $f(k + 1) – mf(k)$ A1: Achieves a correct expression for $f(k + 1) – mf(k)$ in terms of $f(k)$ A1: Reaches a correct expression for $f(k + 1)$ in terms of $f(k)$ A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.

Way 4 B1: Shows that f(1) = 21 M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.) M1: Attempts f(k + 1) A1: Correctly obtains 84M **or** $21\times 5^{2k-1}$ A1: Reaches a correct expression for f(k + 1) in terms of M and 5^{2k-1} A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.

Q7.

Question	Scheme	Marks	AOs
	When $n = 1$, $2^{3n+1}+3\left(5^{2n+1}\right)=16+375=391$ $391=17\times 23$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $2^{3k+1}+3\left(5^{2k+1}\right)$ is divisible by 17	M1	2.4
	$\mathrm{f}(k+1)-\mathrm{f}(k)=2^{3k+4}+3\left(5^{2k+3}\right)-2^{3k+1}-3\left(5^{2k+1}\right)$	M1	2.1
	$=7\times 2^{3k+1}+7\times 3\left(5^{2k+1}\right)+17\times 3\left(5^{2k+1}\right)$		
	$=7\mathrm{f}(k)+17\times 3\left(5^{2k+1}\right)$	A1	1.1b
	$\mathrm{f}(k+1)=8\mathrm{f}(k)+17\times 3\left(5^{2k+1}\right)$	A1	1.1b
	If the statement is true for $n = k$ then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers n.	A1	2.4
		(6)	
			(6 marks)

Q8.

Question	Scheme	Marks	AOs
(a)	$n=1,\ \sum_{r=1}^{1} r^2 = 1$ and $\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(1)(2)(3) = 1$	B1	2.2a
	Assume general statement is true for $n=k$. So assume $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$ is true.	M1	2.4
	$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$	M1	2.1
	$= \frac{1}{6}(k+1)(2k^2+7k+6)$	A1	1.1b
	$= \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)(\{k+1\}+1)(2\{k+1\}+1)$	A1	1.1b
	Then the general result is true for $n=k+1$. As the general result has been shown to be true for $n=1$, then the general result is true for all $n \in \mathbb{Z}^+$.	A1	2.4
		(6)	
(b)	$\sum_{r=1}^{n} r(r+6)(r-6) = \sum_{r=1}^{n} (r^3 - 36r)$		
	$= \frac{1}{4}n^2(n+1)^2 - \frac{36}{2}n(n+1)$	M1 A1	2.1 1.1b
	$= \frac{1}{4}n(n+1)\left[n(n+1)-72\right]$	M1	1.1b
	$= \frac{1}{4}n(n+1)(n-8)(n+9)$ * **cso**	A1*	1.1b
		(4)	
(c)	$\frac{1}{4}n(n+1)(n-8)(n+9) = \frac{17}{6}n(n+1)(2n+1)$	M1	1.1b
	$\frac{1}{4}(n-8)(n+9) = \frac{17}{6}(2n+1)$	M1	1.1b
	$3n^2 - 65n - 250 = 0$	A1	1.1b
	$(3n+10)(n-25) = 0$	M1	1.1b
	(As n must be a positive integer,) $n = 25$	A1	2.3
		(5)	
			(15 marks)

	Question Notes	
(a)	B1	Checks $n=1$ works for both sides of the general statement.
	M1	Assumes (general result) true for $n=k$.
	M1	Attempts to add $(k+1)$th term to the sum of k terms.
	A1	Correct algebraic work leading to either $\frac{1}{6}(k+1)(2k^2+7k+6)$
		or $\frac{1}{6}(k+2)(2k^2+5k+3)$ or $\frac{1}{6}(2k+3)(k^2+3k+2)$
	A1	Correct algebraic work leading to $\frac{1}{6}(k+1)(\{k+1\}+1)(2\{k+1\}+1)$
	A1	cso leading to a correct induction statement conveying **all** three underlined points.
(b)	M1	Substitutes at least one of the standard formulae into their expanded expression.
	A1	Correct expression.
	M1	Depends on previous M mark. Attempt to factorise at least $n(n+1)$ having used both standard formulae correctly.
	A1*	Obtains $\frac{1}{4}n(n+1)(n-8)(n+9)$ by cso.
(c)	M1	Sets their part (a) answer equal to $\frac{17}{6}n(n+1)(2n+1)$
	M1	Cancels out $n(n+1)$ from both sides of their equation.
	A1	$3n^2-65n-250=0$
	M1	A valid method for solving a 3 term quadratic equation.
	A1	**Only one solution** of $n=25$

Q9.

Question	Scheme		Marks	AOs
(i)(a)	$\lvert\mathbf{M}\rvert = 2(1+2) - a(-1-1) + 4(2-1) = 0 \Rightarrow a = ...$		M1	2.3
	The matrix **M** has an inverse when $a \neq -5$		A1	1.1b
			(2)	
(b)	Minors : $\begin{pmatrix} 3 & -2 & 1 \\ -a-8 & 2 & a+4 \\ 4-a & -6 & -2-a \end{pmatrix}$ or Cofactors : $\begin{pmatrix} 3 & 2 & 1 \\ a+8 & 2 & -a-4 \\ 4-a & 6 & -2-a \end{pmatrix}$		B1	1.1b
	$\mathbf{M}^{-1} = \frac{1}{\lvert\mathbf{M}\rvert}\text{adj}(\mathbf{M})$		M1	1.1b
	$\mathbf{M}^{-1} = \frac{1}{2a+10}\begin{pmatrix} 3 & a+8 & 4-a \\ 2 & 2 & 6 \\ 1 & -a-4 & -2-a \end{pmatrix}$	2 correct rows or columns. Follow through their det**M**.	A1ft	1.1b
		All correct. Follow through their det**M**.	A1ft	1.1b
			(4)	

Question	Scheme	Marks	AOs
(ii)	When $n = 1$, lhs $= \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$, rhs $= \begin{pmatrix} 3^1 & 0 \\ 3(3^1-1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ So the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}$	M1	2.4
	$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$	M1	2.1
	$= \begin{pmatrix} 3\times3^k & 0 \\ 3\times3(3^k-1)+6 & 1 \end{pmatrix}$	A1	1.1b
	$= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1}-1) & 1 \end{pmatrix}$	A1	1.1b
	If the statement is true for $n = k$ then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers n.	A1	2.4
		(6)	
			(12 marks)

Notes:
(i)(a) **M1**: Attempts determinant, equates to zero and attempts to solve for a in order to establish the restriction for a **A1**: Provides the correct condition for a if **M** has an inverse
(i)(b) **B1**: A correct matrix of minors or cofactors **M1**: For a complete method for the inverse **A1ft**: Two correct rows following through their determinant **A1ft**: Fully correct inverse following through their determinant
(ii) **B1**: Shows the statement is true for $n = 1$ **M1**: Assumes the statement is true for $n = k$ **M1**: Attempts to multiply the correct matrices **A1**: Correct matrix in terms of k **A1**: Correct matrix in terms of $k + 1$ **A1**: Correct complete conclusion